Math 255B Lecture 19 Notes

Daniel Raban

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1 Development Toward the Spectral Theorem for Unbounded Self-Adjoint Operators

1.1 The resolvent

Let $T : D(T) \to H$ be closed and densely defined. We said $\lambda \notin \mathrm{Spec}(T) \iff T - \lambda : D(T) \to H$ is bijective and defined the **resolvent** as $R(\lambda) = (T - \lambda)^{-1} \in \mathcal{L}(H, H)$ for $\lambda \in \rho(T) = \mathbb{C} \setminus \mathrm{Spec}(T)$, the **resolvent set**.

Proposition 1.1. *The resolvent set $\rho(T) \subseteq \mathbb{C}$ is open, and $\rho(T) \ni \lambda \mapsto R(\lambda) \in \mathcal{L}(H, H)$ is holomorphic.*

Proof. If $\lambda_0 \in \rho(T)$ write

$$T - \lambda = (T - \lambda) - (\lambda - \lambda_0) = (1 - (\lambda - \lambda_0)R(\lambda_0))(T - \lambda_0).$$

It follows that $T - \lambda$ is invertible for $|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0)\|}$. We also have

$$\begin{aligned} R(\lambda) = R(\lambda_0) &= R(\lambda_0)(1 - (\lambda - \lambda_0)R(\lambda_0))^{-1} \\ &= \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R(\lambda_0)^{n+1}. \end{aligned}$$

This converges in $\mathcal{L}(H, H)$, so $R(\lambda)$ is holomorphic. ◻

Proposition 1.2. *If T is self-adjoint, then* $\mathrm{Spec}(T) \subseteq \mathbb{R}$.

Proof. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$\|(T - \lambda)u\|^2 = \|(T - \mathrm{Re}\,\lambda)u\|^2 + (\mathrm{Im}\,\lambda)^2\|u\|^2.$$ ◻

We also have

$$\|R(\lambda)\|_{\mathcal{L}(H,H)} \leq \frac{1}{|\operatorname{Im}\lambda|}.$$

1.2 Nevanlinna-Herglotz functions for self-adjoint operators

Example 1.1. Let $:D(\mathscr{A}) \to H$ be self-adjoint, $\langle \mathscr{A} u, u\rangle \geq c\|u\|^2$ for $u \in D(\mathscr{A})$, and $\mathscr{A}^{-1} : H \to H$ be compact. Then $\operatorname{Spec}(A)$ is of the form $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$. Let e_j be an orthonormal basis of H such that $(\mathscr{A} - \lambda_j)e_j = 0$. Then for $u = \sum_j x_j e_j \in H$,

$$\langle R(z)u, u\rangle = \sum \frac{|c_j|}{\lambda_j - z} = \int \frac{\xi - z}{} d\mu(\xi), \qquad d\mu = \sum |c_j|^2 \delta(\xi - \lambda)$$

This is a positive, pure point measure of total mass $\int d\mu = \|u\|^2$ (by Parseval).

Let $A : D(A) \to H$ be self-adjoint, and consider the holomorphic function

$$f(z) = \langle R(z)u, u\rangle, \qquad \operatorname{Im} z > 0.$$

Then $|f(z)| \leq \frac{C}{|\operatorname{Im} z|}$, where $C = \|u\|^2$. We have

$$2i \operatorname{Im} f(z) = \langle R(z)u, u\rangle - \langle R(z)^* u, u\rangle$$

Using $R(z)^* = R(\overline{z})$,

$$= \langle (R(z) - R(\overline{z}))u, u\rangle$$

We can also check via algebraic manipulation that $R(\lambda) - \mathbb{R}(\mu) = (\lambda - \mu)R(\lambda)R(\mu)$. Using this,

$$= 2i \operatorname{Im} z \, \langle R(z)^* R(z)u, u\rangle .$$

So we get

$$\operatorname{Im} f = \operatorname{Im} z \|R(z)u\|^2.$$

In particular, $\operatorname{Im} f \geq 0$ for $\operatorname{Im} z > 0$.

Now we can use the following general result from complex analysis.[1]

Theorem 1.1 (Nevanlinna, Herglotz,...)**.** *Let f be a holomorphic function in $\operatorname{Im} z > 0$ with $\operatorname{Im} f \geq 0$ and $|f(z)| \leq \frac{c}{\operatorname{Im} z}$. Then there is a uniform bound*

$$\int \operatorname{Im} f(x + iy)\, dx \leq C\pi \qquad \forall y > 0,$$

and there exists a positive bounded measure μ on $\mathbb{R}$ such that

$$\frac{1}{\pi} \int \varphi(x) \operatorname{Im} f(x + iy)\, dx \xrightarrow{y \to 0^+} \int \varphi \, d\mu \qquad \forall \varphi \in C_B := (C \cap L^\infty)(\mathbb{R}).$$

[1] This is a standard result. It was even a qualifying exam problem in the past.

We have

$$f(z) = \int \frac{1}{\xi - z} d\mu(\xi), \qquad \operatorname{Im} z > 0,$$

$$\int d\mu(\xi) = \lim_{y \to +\infty} y \operatorname{Im} f(iy) = \lim_{z \to \infty} (-z f(z)),$$

where $z \to \infty$ with $\arg(z)$ bounded away from $0, \pi$.

Conversely, if $\mu \geq 0$ is a bounded measure on $\mathbb{R}$ and f is defined by $f(z) = \int \frac{1}{\xi - z} d\mu(\xi)$, then both the weak convergence and the limit condition hold.

Proof. Take the semicircle contour γ (slightly above the real axis) made of $\operatorname{Im} z = c > 0$ and an arc of radius R. Cauchy's integral formula gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_\gamma f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) d\zeta, \end{aligned}$$

where z^* is the regkection of z over the line $\operatorname{Im} z = c$. That is, $z^* = \overline{z} + 2ic$.

$$= \frac{1}{2\pi i} \int_\gamma f(\zeta) \frac{z - z^*}{(\zeta - z)(\zeta - z^*)} d\zeta$$

Letting $R \to \infty$, we get

$$f(z) = \frac{1}{2\pi i} \int_{L_c} \frac{f(\zeta)(z - z^*)}{(\zeta - z)(\zeta^* - z^*)} d\zeta,$$

where L_c is the whole line $\operatorname{Im} z = c$ and $(\zeta - z)(\zeta^* - z^*) = |\zeta - z|^2$. Take the imaginary part of this to get

$$\operatorname{Im} f = \frac{\operatorname{Im} z - c}{\pi} \int_{L_c} \frac{\operatorname{Im} f(\zeta)}{|\zeta - z|^2} d\zeta$$

Multiply by $\operatorname{Im} z$ and let $\operatorname{Im} z \to \infty$ while keeping $\operatorname{Re} z$ fixed: the left hand side is $\leq c$. By Fatou's lemma,

$$\frac{1}{\pi} \int_{\mathbb{R}} \operatorname{Im} f(x + ic) \, dx \leq c.$$

By Banach-Alaoglu, there is a sequence of $c_n \to 0^+$ and a positive, bounded measure μ on $\mathbb{R}$ such that

$$\frac{1}{\pi} \operatorname{Im} f(x + ic_n) \, dx \xrightarrow{\text{weak*}} \mu.$$

We get that

$$\operatorname{Im} f(z) = \operatorname{Im} z \int \frac{1}{|\xi - z|^2} d\mu(\xi), \qquad \operatorname{Im} z > 0.$$

This implies that

$$\operatorname{Im}\left(f(z) - \int \frac{1}{\xi - z}\, d\mu(\xi)\right) = 0,$$

so

$$f(z) = \int \frac{1}{\xi - z}\, d\mu(\xi),$$

proving the first claim. $\square$

We will finish the proof of the theorem next time.